HEEGNER POINTS ON ELLIPTIC CURVES WITH A RATIONAL TORSION POINT

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Abstract. Using Heegner points on elliptic curves, we construct points of infinite order on certain elliptic curves with a $\mathbb{Q}$-rational torsion point of odd order. As an application of this construction, we show that for any elliptic curve E defined over $\mathbb{Q}$ which is isogenous to an elliptic curve E' defined over $\mathbb{Q}$ of square-free conductor N with a $\mathbb{Q}$-rational 3-torsion point, a positive proportion of quadratic twists of E have (analytic) rank r, where $r \in \{0, 1\}$. This assertion is predicted to be true unconditionally for any elliptic curve E defined over $\mathbb{Q}$ due to Goldfeld [Go], but previously has been confirmed unconditionally for only one elliptic curve due to Vatsal [V1].

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N and $L(E,s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be its Hasse-Weil L-function. Let $X_0(N)$ be the modular curve of level N with Jacobian $J = J_0(N)$. It is known that there exists a newform $f_E(z) = \sum_{n=1}^{\infty} a(n)q^n$ of level N and a morphism $\phi : X_0(N) \to E$ defined over $\mathbb{Q}$. This morphism factors in $J_0(N)$ through a homomorphism $\pi : J_0(N) \to E$. Let $\pi^* : E \to J_0(N)$ be its dual map.

Definition 1.1. A $\mathbb{Q}$-rational torsion point P of order l on E is *cuspidal* if $\pi^*(P)$ is a $\mathbb{Q}$-rational cuspidal divisor of order l in $J_0(N)$.

Using Heegner points on $X_0(N)$, Birch [Bi] constructed points of infinite order on certain elliptic quotients of $J_0(N)$ with a $\mathbb{Q}$-rational cuspidal torsion point of even order. Later, Gross [Gr] and Mazur [Ma] developed some methods to construct points of infinite order in Eisenstein quotients of $J_0(N)$, when N is prime.

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In this paper, using techniques similar to those developed by Birch [Bi], Gross [Gr], and Mazur [Ma], we shall construct points of infinite order on certain elliptic curves with a $\mathbb{Q}$-rational cuspidal torsion point of odd order. (See Section 2.)

As an application of this construction, we have the following theorem.

Theorem 1.2. *Let $r \in \{0,1\}$. Let E be an elliptic curve defined over $\mathbb{Q}$ of square-free conductor N with a $\mathbb{Q}$-rational cuspidal 3-torsion point. Then*

$$\sharp\{|D_F| < X| \ \mathrm{Ord}_{s=1} L(E_{D_F}, s) = r\} \gg X,$$

where D_F is a fundamental discriminant of quadratic field F and E_{D_F} is the quadratic twist of E.

We do not know yet that every $\mathbb{Q}$-rational torsion point of an elliptic curve is *cuspidal*. But, in the proof of Proposition 5.3 of [V3], Vatsal showed that if E' is an elliptic curve defined over $\mathbb{Q}$ of conductor N such that $l^2 \nmid N$ with a $\mathbb{Q}$-rational l-torsion point, then the optimal elliptic curve E in the isogeny class of E' has a $\mathbb{Q}$-rational *cuspidal* l-torsion point. And we note that if two elliptic curves E' and E are in the same isogeny class, then $L(E', s) = L(E, s) = \sum_{n=1}^{\infty} a(n)n^{-s}$. So if D_F is coprime to the conductor of E, then

$$L(E'_{D_F}, s) = L(E_{D_F}, s) = \sum_{n=1}^{\infty} \chi_D(n) a(n) n^{-s},$$

where $\chi_{D_F} = (\frac{D_F}{\cdot})$ denote the usual Kronecker character. Thus from Theorem 1.2, we immediately have the following theorem.

Theorem 1.3. *Let $r \in \{0,1\}$. Let E be an elliptic curve defined over $\mathbb{Q}$ which is isogenous to an elliptic curve E' defined over $\mathbb{Q}$ of square-free conductor N with a $\mathbb{Q}$-rational 3-torsion point. Then*

$$\sharp\{|D_F| < X| \ \mathrm{Ord}_{s=1} L(E_{D_F}, s) = r\} \gg X,$$

where D_F is a fundamental discriminant of quadratic field F and E_{D_F} is the quadratic twist of E.

This assertion is predicted to be true for any elliptic curve E defined over $\mathbb{Q}$ by the the following conjecture.

Conjecture (Goldfeld [Go]). *For any elliptic curve E defined over $\mathbb{Q}$,*

$$\sum_{|D_F|<X} \mathrm{Ord}_{s=1} L(E_{D_F}, s) \sim \frac{1}{2} \sum_{|D_F|<X} 1.$$

We note that this conjecture is proved by Iwaniec and Sarnak [I-S] under assumption of the Riemann hypothesis.

The first unconditional example $X_0(19)$ is known by Vatsal [V1]. In [V2], Vatsal proved that if E be a semi-stable elliptic curve defined over $\mathbb{Q}$ with a $\mathbb{Q}$-rational point of order 3 and good reduction at 3, for a positive proportion of D_F, $\mathrm{Ord}_{s=1} L(E_{D_F}, s) = 0$. Recently, we [B-J-K] constructed infinitely many elliptic curves E defined over $\mathbb{Q}$ such that for a positive proportion of D_F, $\mathrm{Ord}_{s=1} L(E_{D_F}, s) = 1$.

2. Heegner points on elliptic curves

Let K be an imaginary quadratic field with the fundamental discriminant D_K. For a square-free positive integer c, let $\mathcal{O}$ be the order in K with conductor c and discriminant $D = D_K c^2$. Suppose that $D = B^2 - 4NC$ has integer solutions with $\gcd(N, B, C) = 1$. Following [Gr], we denote the *Heegner points* on $X_0(N)$ with coordinates $(j(\mathfrak{a}), j(\mathfrak{n}\mathfrak{a}))$ by

$$(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]),$$

where $\mathfrak{n}$ is a degree-one factor of N in K, $\mathfrak{a}$ is an invertible ideal of $\mathcal{O}$ and $[\mathfrak{a}]$ is its ideal class. The Heegner point $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$ has coordinates in the ring class field H corresponding to $\mathcal{O}$. It is known that $\mathrm{Gal}(H/K)$ is isomorphic to the group $\mathrm{Pic}(\mathcal{O})$ of invertible ideal classes of $\mathcal{O}$.

Let $D = D_K c^2 = d_1 d_2$, where $d_1 > 0$. $d_2 < 0$ are fundamental discriminants and χ be the corresponding ring class character of $\mathrm{Gal}(H/K)$, which is given by

$$\chi(\mathfrak{b}) = \chi_{d_1}(\mathrm{N}\mathfrak{b}) = \chi_{d_2}(\mathrm{N}\mathfrak{b}),$$

for an ideal $\mathfrak{b}$ prime to D. Here, χ_{d_1}, χ_{d_2} are the quadratic characters corresponding to the real quadratic field $k_1 = \mathbb{Q}(\sqrt{d_1})$ and the imaginary quadratic field $k_2 = \mathbb{Q}(\sqrt{d_2})$. Let h_1 and h_2 be the class numbers of k_1 and

k_2 respectively. Then the biquadratic field $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ is the fixed field of $\ker\chi$ on H.

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N and $\phi : X_0(N) \to E$ be a morphism defined over $\mathbb{Q}$. If $\chi \neq 1$, we define

$$P_E(d_1, d_2) := \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a})\phi((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])).$$

Then we have

$$P_E(d_1, d_2) \in E(L)^{\chi},$$

where $E(L)^{\chi}$ is the minus eigenspace for the action of the group $\mathrm{Gal}(L/K)$ on $E(L)$. For more detail, see [B-S] and [Gr].

Now we recall the following work of Ligozat [Li]; if D_0 is a $\mathbb{Q}$-rational cuspidal divisor of order l in $J_0(N)$, then there is a *Dedekind eta-product*

$$g_{\mathbf{r}} := \prod_{d|N} \eta_d^{r_d}$$

which is a modular function on $X_0(N)$ defined over $\mathbb{Q}$ and satisfies

$$\mathrm{div}\ g_{\mathbf{r}} = lD_0.$$

Here $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function and $\eta_d(z) := \eta(dz)$. The family of integers $\mathbf{r} = (r_d)$ indexed by all the positive divisors d of N should satisfy the following four conditions; (i) $\sum_{d|N} r_d = 0$, (ii) $\sum_{d|N} d\, r_d \equiv 0 \mod 24$, (iii) $\sum_{d|N} \frac{N}{d} r_d \equiv 0 \mod 24$, (iv) $\prod_{d|N} d^{r_d}$ is the square of a rational number. Then we can state the following theorem.

Theorem 2.1. *Let $l \in \{3, 5, 7\}$. Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N with a $\mathbb{Q}$-rational cuspidal l-torsion point P. Let $g_{\mathbf{r}} := \prod_{d|N} \eta_d^{r_d}$ be the Dedekind eta-product satisfying* $\mathrm{div}\ g_{\mathbf{r}} = l\pi^*(P)$*. Let w be the number of units in the imaginary quadratic field $\mathbb{Q}(\sqrt{d_2})$. Assume that* $\gcd(l, w) = 1$ *and the quadratic twists E_{d_1}, E_{d_2} have no $\mathbb{Q}$-rational l-torsion points. For $d|N$, let $(d) = \mathfrak{d}\bar{\mathfrak{d}}$ in $\mathcal{O}$. If $\chi \neq 1$ and $l \nmid h_1h_2(\sum_{d|N}\chi(\mathfrak{d})r_d)$, then $P_E(d_1, d_2)$ is of infinite order in $E(L)^{\chi}$.*

Let

$$L(E, \chi, s) := L(E_{d_1}, s)L(E_{d_2}, s).$$

By the work of Gross and Zagier [G-Z] for a square-free D and the work of Zhang [Zh] for a general D, we know that if $P_E(d_1, d_2)$ is of infinite order in $E(L)^\chi$, $L'(E, \chi, 1)$ does not vanish. By the functional equation satisfied by each of the factors of $L(E, \chi, s)$, we have the following corollary.

Corollary 2.2. *Assume that an elliptic curve E satisfies the same condition of Theorem 2.1. Let ϵ be the sign of the functional equation of $L(E, s)$. If $\epsilon\chi_{d_1}(-N) = -\epsilon\chi_{d_2}(-N) = 1$,*

$$L(E_{d_1}, 1) \neq 0 \;\; \textit{and} \;\; L'(E_{d_2}, 1) \neq 0,$$

and if $\epsilon\chi_{d_1}(-N) = -\epsilon\chi_{d_2}(-N) = -1$,

$$L'(E_{d_1}, 1) \neq 0 \;\; \textit{and} \;\; L(E_{d_2}, 1) \neq 0.$$

In [B-J-K], we obtained a similar result for the case of $\chi = 1$ under the condition that $\prod_{d|N} d^{r_d} \neq \alpha^l$ for any $\alpha \in \mathbb{Q}$. But this case is so restricted that we can construct only infinitely many elliptic curves E with a positive portion of rank-one quadratic twists. A main ingredient in the proof of Theorem 2.1 is Kronecker's limit formula.

3. Kronecker's limit formula

Let $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$ be the Heegner point on $X_0(N)$. We can choose an oriented basis $<\omega_1, \omega_2>$ of $\mathfrak{a}$ such that $\mathfrak{a}\mathfrak{n}^{-1} =<\omega_1, \omega_2/N>$. Let $\tau = \omega_1/\omega_2$. Then $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$ is the $\Gamma_0(N)$ orbit of τ. Since $\tau \in K$, τ satisfies an integral quadratic equation $A\tau^2 + B\tau + C = 0$ with $A > 0$ and $\gcd(A, B, C) = 1$. We note that $D = B^2 - 4AC$, $A = NA'$ and $\gcd(N, B, A'C) = 1$. Let $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ be the binary positive definite quadratic form.

To a positive definite binary quadratic form Q, we associate the zeta function

$$\zeta_Q(s) := \sum_{m,n=-\infty}^{\infty} Q(m, n)^{-s}.$$

Let $\mathbf{r} = (r_d)$ is a family of integers $r_d \in \mathbb{Z}$ indexed by all the positive divisors d of N such that $\sum_{d|N} r_d = 0$. Let $\Delta(z) := \eta(z)^{24}$ and $\Delta_d(z) := \Delta(dz)$. The *modular unit* $\Delta_\mathbf{r}(z) := \prod_{d|N} \Delta(dz)^{r_d}$ is a modular function on $X_0(N)$ defined over $\mathbb{Q}$. We define

$$\zeta(\Delta_\mathbf{r}, \tau, s) := \sum_{d|N} r_d d^{-s} \zeta_{d\tau}(s),$$

where $\zeta_{d\tau}(s) := \zeta_{Q_{d\tau}}(s)$ and $Q_{d\tau} = \frac{A}{d}x^2 + Bxy + dCy^2$.

The first Kronecker limit formula implies that the function $\zeta(\Delta_{\mathbf{r}}, \tau, s)$ is holomorphic except for a simple pole at $s = 1$, vanishes at $s = 0$ and

$$\zeta'(\Delta_{\mathbf{r}}, \tau, 0) = -\frac{1}{12}\log|\Delta_{\mathbf{r}}(\tau)|$$

[Theorem 1.2, D-D].

Proposition 3.1. *Let $L(\chi, s)$ be the abelian L-function of the character $\chi \neq 1$. For $d|N$, let $(d) = \mathfrak{d}\bar{\mathfrak{d}}$ in $\mathcal{O}$ and $S := \sum_{d|N} \chi(\mathfrak{d}) r_d$. Then*

$$SL'(\chi, 0) = -\frac{1}{12}\log|\prod_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \Delta_{\mathbf{r}}(\tau)^{\chi(\mathfrak{a})}|.$$

Proof: Since $\zeta(\Delta_{\mathbf{r}}, \tau, s) := \sum_{d|N} r_d d^{-s} \zeta_{d\tau}(s)$,

$$\zeta'(\Delta_{\mathbf{r}}, \tau, s) = \sum_{d|N} r_d d^{-s} \zeta'_{d\tau}(s) - \sum_{d|N} r_d d^{-s} (\log d) \zeta_{d\tau}(s).$$

Since $\zeta_{d\tau}(0) = -1$ for all $d|N$,

$$\zeta'(\Delta_{\mathbf{r}}, \tau, 0) = \sum_{d|N} r_d \zeta'_{d\tau}(0) + \sum_{d|N} r_d \log d.$$

So

$$\begin{aligned}
&\sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \zeta'(\Delta_{\mathbf{r}}, \tau, 0) \\
=& \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \sum_{d|N} r_d \zeta'_{d\tau}(0) + \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \sum_{d|N} r_d \log d \\
=& \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \sum_{d|N} r_d \zeta'_{d\tau}(0) \\
=& \sum_{d|N} \chi(\mathfrak{d}) r_d \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{d}\mathfrak{a}) \zeta'_{d\tau}(0) \\
=& S \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{d}\mathfrak{a}) \zeta'_{d\tau}(0).
\end{aligned}$$

Since $L(\chi, s) = \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \zeta_{\tau}(s)$,

$$\begin{aligned}
SL'(\chi, 0) &= \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \chi(\mathfrak{a}) \zeta'(\Delta_{\mathbf{r}}, \tau, 0) \\
&= -\frac{1}{12}\log|\prod_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O})} \Delta_{\mathbf{r}}(\tau)^{\chi(\mathfrak{a})}|.
\end{aligned}$$

□

4. Proof of Theorem 2.1

In this section, applying the method developed by Gross in [Gr] to the elliptic curve E in Theorem 2.1, we shall prove Theorem 2.1.

Let $l \in \{3, 5, 7\}$. Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N with a $\mathbb{Q}$-rational cuspidal l-torsion point P. Let $f \in \mathbb{Q}(E)$ such that $\operatorname{div} f = lP$. Then f induces a homomorphism

$$\delta : E(L)/lE(L) \to L^*/(L^*)^l.$$

In particular, when $\chi \neq 1$, we have

$$\delta(P_E(d_1, d_2)) = \prod_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})} f(\phi((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])))^{\chi(\mathfrak{a})}.$$

Let $g_{\mathbf{r}} := \prod_{d|N} \eta_d^{r_d}$ be the Dedekind eta-product satisfying $\operatorname{div} g_{\mathbf{r}} = l\pi^*(P)$. Since $\operatorname{div}(f \circ \phi) = l(\pi^*(P) + \operatorname{div} g)$ for some $g \in \mathbb{Q}(X_0(N))$, we have

$$f \circ \phi = \alpha \cdot g_{\mathbf{r}} \cdot g^l$$

for some constant $\alpha \in \mathbb{Q}$. Thus

$$\begin{aligned} \delta(P_E(d_1, d_2)) &= \beta^l \cdot \prod_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})} g_{\mathbf{r}}((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]))^{\chi(\mathfrak{a})} \\ &= \beta^l \cdot \prod_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})} \Delta_{\mathbf{r}}((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]))^{\frac{\chi(\mathfrak{a})}{24}}, \end{aligned}$$

for some $\beta \in L$. Let $E_\chi := \prod_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})} \Delta_{\mathbf{r}}((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]))^{\frac{\chi(\mathfrak{a})}{24}}$. Then

$$\delta(P_E(d_1, d_2)) \equiv E_\chi \pmod{(L^*)^l}.$$

By Proposition 3.1,

$$\log |E_\chi| = -\frac{S}{2} L'(\chi, 0).$$

The L-function factors as $L(\chi, s) = L(\chi_{d_1}, s)L(\chi_{d_2}, s)$. Since $L'(\chi_{d_1}, 0) = h_1 \log u$ and $L(\chi_{d_2}, 0) = 2h_2/w$, we have

$$\log |E_\chi| = (h_1 h_2 S/w) \log u,$$

where u is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d_1})$ and w is the number of units in the imaginary quadratic field $\mathbb{Q}(\sqrt{d_2})$. Hence

$$E_\chi = \zeta \cdot u^{h_1 h_2 S/w},$$

where ζ is a root of unity in $(L^*)^\chi$. Since $(l, w) = 1$, ζ is a lth-power and $\delta(P_E(d_1, d_2))$ is a lth-power if and only if $l | h_1 h_2 S$. So if $l \not| h_1 h_2 S$, then $\delta(P_E(d_1, d_2))$ is nontrivial in $L^*/(L^*)^l$. Since $E(L)^\chi = E(k_1)^{\chi_{d_1}} \bigoplus E(k_2)^{\chi_{d_2}}$, if the quadratic twists E_{d_1}, E_{d_2} have no $\mathbb{Q}$-rational l-torsion points, then $P_E(d_1, d_2)$ is of infinite order in $E(L)^\chi$. □

5. Proof of Theorem 1.2

In this section, using the method developed by Vatsal in [V1], we shall prove Theorem 1.2. A new ingredient in this proof is using an indivisibility property of class numbers of quadratic fields in [By] instead of Scholz's reflection theorem used in [V1].

Let N be a square-free positive integer. Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N with a $\mathbb{Q}$-rational cuspidal 3-torsion point P. Let ϵ be the sign of the functional equation of $L(E, s)$. Let $t \equiv 3 \pmod 4$ be a positive square-free integer such that every prime $p | N$ splits in $K = \mathbb{Q}(\sqrt{-t})$ with $D_K = -t$. Let $c \equiv 1 \pmod 4$ be a positive square-free integer such that $\gcd(c, tN) = 1$. Let $\chi \neq 1$ be the ring class character of K which is determined by a factorization

$$D = D_K c^2 = d_1 d_2,$$

where $d_1 = c$ and $d_2 = D_K c = -tc$. Then

$$D = -tc^2 = B^2 - 4NC$$

has integer solutions with $\gcd(N, B, C) = 1$. So we can define the Heegner point $P_E(d_1, d_2) \in E(L)^\chi$.

Let $g_{\mathbf{r}} := \prod_{d|N} \eta_d^{r_d}$ be the Dedekind eta-product satisfying div $g_{\mathbf{r}} = 3\pi^*(P)$. Since N is square-free, if $3 \,|\, \sum_{d|N} \chi(\mathfrak{d}) r_d$ for all $\chi \neq 1$, then $3 \,|\, 2^s r_d$ for all $d|N$, where s is the number of different prime factors of N. In this case, $3 | r(d)$ for all $d|N$ and $\pi^*(P)$ should be trivial. But it is impossible. So we can always choose $\chi \neq 1$ such that $3 \not| \sum_{d|N} \chi(\mathfrak{d}) r_d$.

By a theorem of Davenport and Heilbronn [D-H], which is refined by Nakagawa and Horie [N-H] and a theorem of the author [By], which also can be easily refined as the form of congruence class in [N-H], we know that a positive proportion of positive square-free integers c satisfies the following conditions for a fixed t and a fixed $\chi \neq 1$ such that $3 \nmid \sum_{d|N} \chi(\mathfrak{d}) r_d$;

(i) $c \equiv 1 \pmod 4$ and $\gcd(c, tN) = 1$,

(ii) $\chi_c(d) = \chi(\mathfrak{d})$ for all $d|N$,

(iii) $3 \nmid h_1 h_2$,

where h_1 is the class number of the real quadratic field $\mathbb{Q}(\sqrt{c})$ and h_2 is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-tc})$. We note that for only finitely many $d_1 = c$ and $d_2 = -tc$, E_{d_1} and E_{d_2} have $\mathbb{Q}$-rational 3-torsion points and $\gcd(3, w) = 1$.

Thus by Corollary 2.2, if $\epsilon\chi(\mathfrak{n}) = 1$, for a positive proportion of positive square-free integers c, $\mathrm{Ord}_{s=1} L(E_c, s) = 0$ and $\mathrm{Ord}_{s=1} L(E_{-tc}, s) = 1$. Similarly if $\epsilon\chi(\mathfrak{n}) = -1$, for a positive proportion of positive square-free integers c, $\mathrm{Ord}_{s=1} L(E_c, s) = 1$ and $\mathrm{Ord}_{s=1} L(E_{-tc}, s) = 0$. Hence we completed the proof of Theorem 1.2. □

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